Coisotropic deformation of an elliptic curve and linear flows on the first Birkhoff stratum

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Introduction

In \cite{3} the authors remark the relevance of the coisotropic deformations of associative commutative algebras in the context of the dispersionless hierarchies of integrable systems. The starting point of this approach is an \textit{n}\textsuperscript{-}generated polynomial algebra $\mathbb{C}[p_0, \ldots, p_{n-1}]$ and an ideal $\mathcal{J}_0 \subset \mathbb{C}[p_0, \ldots, p_{n-1}]$. Let us next consider a deformation $\mathcal{A}_n^u(p)$ of the algebra given by an \textit{n}\textsuperscript{-}generated algebra of polynomials in $p_0, \ldots, p_{n-1}$ whose coefficients depend on the parameters $x_0, \ldots, x_{n-1}$. In the same way let us also take a deformation $\mathcal{J}_n^u(p)$ of the ideal $\mathcal{J}_0$.

The key point in the definition of coisotropic deformation is considering the elements of $\mathcal{A}_n^u(p)$ as functions of a suitable Poisson space, in our cases $\mathbb{R}^{2n}$ with co-ordinates $(x_i, p_j)$ and equipped with the canonical structure $\{x_i, p_j\} = \delta_{ij}$. The algebra $\mathcal{A}_n^u(p) \setminus \mathcal{J}_n^u(p)$ is a coisotropic deformation of $\mathbb{C}[p_0, \ldots, p_{n-1}] \setminus \mathcal{J}_0$ if

$$\{\mathcal{J}_n^u(p), \mathcal{J}_n^u(p)\} \subset \mathcal{J}_n^u(p).$$

In this setup the dispersionless Kadomtsev-Petviashvili equation is related to a coisotropic deformation \cite{3} of the polynomial algebra $\mathbb{C}[p_1, p_2, p_3]$ with the ideal $J_{KP} = \langle p_3 - p_1^3 - u_1p - u_0, p_2 - p^2 - v_0 \rangle$. If we introduce the deformation $u_i = u_i(x_1, x_2, x_3)$ for $i = 0, 1$ and $v_0 = v_0(x_1, x_2, x_3)$, then the condition

$$\{p_3 - p_1^3 - u_1p_1 - u_0, p_2 - p_1^2 - v_0\} \in J_{KP}$$

gives rise to the dKP equation. Other similar 2 + 1D integrable systems such as n-dKP, can be interpreted as coisotropic deformations of a polynomial algebra quotiented by an ideal generated by algebraic curves of genus zero.

Our results

In \cite{5} we extend this approach taking into account ideals generated by genus 1 algebraic curves. We start from $\mathbb{C}[p_2, p_3, p_4]$ and the ideal $\mathcal{J}_1 = \langle \mathcal{E}, J^{(4)} \rangle$, where

$$J^{(4)} = p_4 - p_2^2 - v_2p_3 - v_1p_2 - v_0$$
and

\[ E = p_3^2 - p_2^2 - u_4p_3p_2 - u_3p_2^2 - u_2p_3 - u_1p_2 - u_0 \]

is a generic elliptic curve. We show that the coisotropy conditions

\[ \{E, J^{(4)}\} \in \mathcal{J}_1, \]

applied to the algebra deformation depending on the parameters \(x_2, x_3, x_4,\) are equivalent to the dispersionless counterpart of the 2 + 1D linear flow on the first Birkhoff stratum of the Universal Sato Grassmannian (see for this system [1, 4]).

In general, we conjecture that the dispersionless counterpart of any Grassmannian integrable flow [7, 2] are deformed associative algebras. The complete classification of the algebras and the related tau structure is a work in progress.

In the literature, e.g. in [6], the authors show that the dispersionless 3 component KdV equation can be interpreted as a 1 + 1D deformation of an elliptic curve: Such deformation is a particular reduction of the 2 + 1D system obtained from (2).

The extension of all the previous results to the whole hierarchy implies the use of an infinitely many generated associative algebra.

References


