Differential operators on the space of periodic functions

M. Maldonado\textsuperscript{a} J. Prada\textsuperscript{a} M. J. Senosiain\textsuperscript{a}

March 1, 2009

NEEDS 2009 Workshop
Nonlinear Evolution Equations and Dynamical Systems

\textsuperscript{a} University of Salamanca, Spain.

The problem of the equivalence of linear differential operators, notion first introduced by Delsarte (1938), is a long standing one. Two operators $A$ and $B$ on a space $H$ are equivalent if there is an isomorphism $X$ such that $AX =XA$.

On the space $C^\infty_{2\pi}(\mathbb{R})$ of all $2\pi$–periodic $C^\infty$–functions on $\mathbb{R}$, we consider two finite linear differential operators $T_1 = \sum_{k=1}^{m} a_k D^k$ and $T_2 = \sum_{k=1}^{m} b_k D^k$, where $a_k, b_k$ are either constants or exponential functions $e^{ipx}, p \in \mathbb{Z}$.

It is well-known that the space $C^\infty_{2\pi}(\mathbb{R})$ with its natural topology given by the seminorms

$$||f||_k^2 = \sum_{0 \leq p \leq k} ||f^p||^2_{L^2[-\pi, \pi]}$$

is isomorphic to the sequence space $s$ of rapidly decreasing sequences

$$s = \{x = (x_n) : \lim |x_j|^k = 0, \text{ for all } k \in \mathbb{N} \}$$

with its natural topology given by the seminorms

$$||x||_k = \sum_{j \in \mathbb{N}} |x_j|^k \text{ for all } k \in \mathbb{N}$$

that is the mapping

$$f \overset{F}{\rightarrow} (f_0, f_1, f_{-1}, f_2, f_{-2}, ....)$$

is a linear bijection, where

$$f(x) = \sum_{n \in \mathbb{Z}} f_n e^{inx}$$

is the Fourier series of $f$ [6].

The operators $T_1$ and $T_2$ induce two linear operators $A_1$ and $A_2$ from $s$ to $s$, in fact

$$A_1 = FT_1 F^{-1} \quad A_2 = FT_2 F^{-1}$$

The problem addressed in this work is the following one: Is there an isomorphism $S$ such that

$$SA_1 = A_2 S$$
and consequently
\[ F^{-1}SF T_1 = T_2 F^{-1}SF \]
which implies that the operators \( T_1 \) and \( T_2 \) are equivalent?

The answer to this problem in our particular case seems to be negative, at least the examples we present do not have a solution. We intend to go more deeply into the problem in future works.

References