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Reciprocal transformations and local Hamiltonian structures of hydrodynamic type systems

Simonetta Abenda[a]

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a. Department of Mathematics, University of Bologna, Bologna, Italy.

Systems of hydrodynamic type are quasilinear evolutionary hyperbolic PDEs of the form

$$u^i_t = \sum_{k=1}^n v^i_k(u) u^k_x, \qquad u = (u^1, \dots, u^n), \quad u^i_x = \frac{\partial u^i}{\partial x}, \quad u^i_t = \frac{\partial u^i}{\partial t}. \tag{1}$$

They naturally arise in applications such as gas dynamics, hydrodynamics, chemical kinetics, the Whitham averaging procedure, differential geometry and topological field theory. Dubrovin and Novikov showed that (1) is a local Hamiltonian system (DN system) with Hamiltonian $H[u] = \int h(u)dx$, if there exists a flat non-degenerate metric tensor $g(u)$ in $\mathbb{R}^n$ with Christoffel symbols $\Gamma^i_{jk}(u)$, such that the matrix $v^i_k(u)$ can be represented in the form

$$v^i_k(u) = \sum_{k=1}^n \left(g^{il}(u) \frac{\partial^2 h}{\partial u^l \partial u^k}(u) - \sum_{s=1}^n g^{ik}(u) \Gamma^l_{sk}(u) \frac{\partial h}{\partial u^l}(u) \right).$$

In this talk I shall consider DN systems which possess Riemann invariants, *i.e.* they may be transformed to the diagonal form $u^i_t = v^i(u^1, \dots, u^n) u^i_x$, $i = 1, \dots, n$, and are integrable. I shall present some recent results of mine (J.Phys. A, vol. 42 (2009)), where I settle the necessary conditions on the conservation laws in the reciprocal transformation so that, after such a transformation of the independent variables, one of the metrics associated to the initial system be flat.

Reciprocal transformations change the independent variables of a system and are an important class of nonlocal transformations which act on hydrodynamic–type systems. Such transformations map conservation laws to conservation laws and map diagonalizable systems to diagonalizable systems, but act non trivially on the metrics and on the Hamiltonian structures: for instance, the flatness property for metrics as well as the locality of the Hamiltonian structure are not preserved, in general, by such transformations. Then, it is natural to investigate under which additional hypotheses the reciprocal system still possesses a local Hamiltonian structure, our ultimate goal being the search for new examples of integrable Hamiltonian systems and the geometrical characterization of the associated hypersurfaces.

With this in mind, we start from a smooth integrable Hamiltonian system in Riemann invariant form

$$u^i_t = v^i(u) u^i_x, \qquad i = 1, \dots, n, \tag{2}$$

with smooth conservation laws $B(u)_t = A(u)_x$, $N(u)_t = M(u)_x$ such that $B(u)M(u) - A(u)N(u) \neq 0$. In the new independent variables $\hat{x}$ and $\hat{t}$ defined by

$$d\hat{x} = B(u)dx + A(u)dt, \qquad d\hat{t} = N(u)dx + M(u)dt, \tag{3}$$

the reciprocal system is still diagonal, the metric of the initial systems $g_{ii}(u)$ transforms to

$$\hat{g}_{ii}(u) = \left(\frac{M(u) - N(u)v^i(u)}{B(u)M(u) - A(u)N(u)} \right)^2 g_{ii}(u), \tag{4}$$

and all conservation laws and commuting flows of the original system (2) may be recalculated in the new independent variables.

In particular, we prove the following statement: let $n \geq 3$ in the case of reciprocal transformations of a single independent variable or $n \geq 5$ in the case of transformations of both the independent variable; then the reciprocal metric may be flat only if the conservation laws in the transformation are linear combinations of the canonical densities of conservation laws, *i.e* the Casimirs, the momentum and the Hamiltonian densities associated to the Hamiltonian operator for the initial metric.

Then, we restrict ourselves to the case in which the initial metric is either flat or of constant curvature and we classify the reciprocal transformations of one or both the independent variables so that the reciprocal metric is flat.

Theorem Let $n \geq 5$ and let $u^i_t = v^i(u)u^i_x$, $i = 1, \ldots, n$, be a DN strictly hyperbolic hydrodynamic type system and $g_{ii}(u)$ the flat metric of its Hamiltonian operator. Let $d\hat{x} = B(u)dx + A(u)dt$, $d\hat{t} = N(u)dx + M(u)dt$ be a reciprocal transformation with $A(u), B(u), M(u)$ and $N(u)$ not all constant functions.

Then the reciprocal metric $\hat{g}_{ii}(u)$ is flat if and only if one of the following alternatives hold:

A.i) there exist constants $\kappa_1 \neq 0, \kappa_2, \kappa_3$ such that

$$M(u) = \kappa_1, \qquad N(u) = \kappa_2, \qquad (\nabla B)^2(u) = \kappa_3 \left(\kappa_1 B(u) - \kappa_2 A(u)\right);$$

A.ii) there exist constants $\kappa_1 \neq 0, \kappa_2, \kappa_3$ such that

$$B(u) = \kappa_1, \qquad A(u) = \kappa_2, \qquad (\nabla N)^2(u) = \kappa_3 \left(\kappa_1 M(u) - \kappa_2 N(u)\right);$$

A.iii) there exist constants $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ such that

$$(\nabla B)^2(u) = 2\kappa_1 A(u) + 2\kappa_2 B(u), \qquad (\nabla N)^2(u) = 2\kappa_3 M(u) + 2\kappa_4 N(u),$$

$$< \nabla B(u), \nabla N(u) >= \kappa_1 M(u) + \kappa_2 N(u) + \kappa_3 A(u) + \kappa_4 B(u). \quad \square$$

Such characterization has an interesting geometric interpretation in view of previous results by E.V. Ferapontov.

Corollary Let $n \geq 5$. The hypersurfaces associated to two diagonalizable strictly hyperbolic DN systems are connected by a Lie sphere transformation if and only if the local Hamiltonian structures of the two DN systems are connected by canonical reciprocal transformation satisfying the above Theorem. $\square$