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Solitons, boundary value problems and a nonlinear method of images

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One of the hallmarks of integrability is the existence of exact N-soliton solutions. It is well known that each soliton is associated to a discrete eigenvalue for the scattering problem via the inverse scattering transform (IST). This has long been known to be true for initial value problems (IVPs) posed on an infinitely extended spatial domain. Recent developments on the IST for initial-boundary value problems (IBVPs), however, have shown that the same statement also applies for problems posed over a semi-infinite interval.

Here we characterize the soliton solutions of IBVPs for the focusing nonlinear Schrödinger (NLS) equation

$$iq_t + q_{xx} + 2\nu|q|^2 q = 0\,. \tag{1}$$

The well-known one-soliton solution of (1) is

$$q_{\rm s}(x,t) = A\mathrm{e}^{i[Vx+(A^2-V^2)t+\varphi]}\,\mathrm{sech}[A(x-2Vt-\xi)]\,, \tag{2}$$

where $k=(V+i\,A)/2$ is the discrete eigenvalue. The IBVP for (1) on $0<x<\infty$ with homogeneous Dirichlet or Neumann boundary conditions (BCs) at the origin was studied in [1] using the IST on the whole line and an odd or even extension of the potential, respectively. The case of homogeneous Robin BCs,

$$q_x(0,t) - \alpha q(0,t) = 0\,, \tag{3}$$

with $\alpha\in\mathbb{R}$, was also studied in [2] using a clever extension of the potential to the whole line. Finally, a new spectral method was recently proposed for the solution of IBVPs for integrable NLEEs [3].

Importantly, in all of these methods the relation between solitons and discrete eigenvalues existing in the IVP is preserved in the IBVP. This leads to a paradox, however, since (2) does not satisfy the BCs (3). A further paradox is that numerical solutions of the IBVP for (1) show inequivocally that solitons are reflected at the boundary. But the soliton velocity is the real part of the discrete eigenvalue, which does not change in time. The resolution of these apparent paradoxes is that discrete eigenvalues in the IBVP appear in *quartets*, as opposed to pairs in the IVP. This means that, for each soliton in the physical domain (in our case, the positive x-axis), a symmetric counterpart exists (i.e., the negative x-axis), with equal amplitude and opposite velocity, whose presence ensures that the whole solution satisfies the BCs. The ostensible reflection of the soliton at the boundary of the physical domain then corresponds simply to the interchanging of roles between the "physical" and "mirror" solitons.

The method to obtain soliton solutions for the IBVP on the half line is similar in spirit to the method of images that is used to solve boundary value problems in electrostatics. Here, however, the reflection experienced by the solitons comes accompanied by a corresponding position shift, which is a reminder of the nonlinear nature of the problem. We also show that the reflection-induced shift has two components, one being the position shift experienced in any soliton interaction and the second related to the location of the mirror soliton.

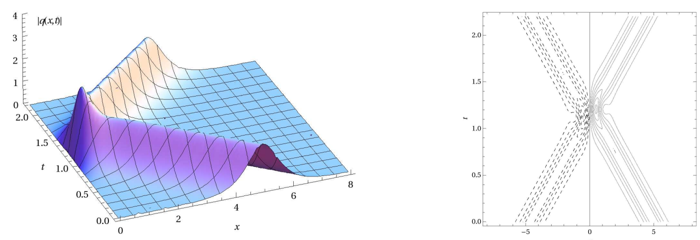

Figure 1: Soliton reflection with Dirichlet BCs, with $A = 2$, $V = -2$, $\xi = 5$, and $\varphi = 0$. Left: three-dimensional (3D) plot of $|q(x,t)|$; right: contour plot of $|q(x,t)|$ showing the mirror soliton (dashed) to the left of the boundary.

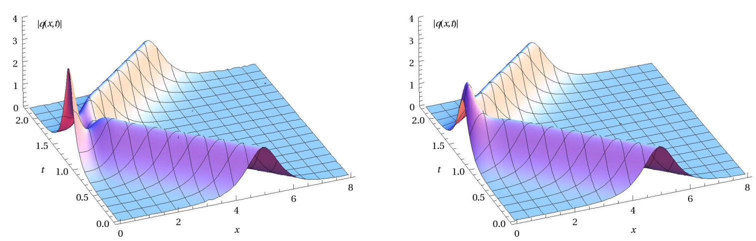

Figure 2: Soliton reflection with Neumann BCs (left) and Robin BCs with $\alpha = 3$ (right), both with $A = 2$, $V = -2$, $\xi = 5$, and $\varphi = 0$.

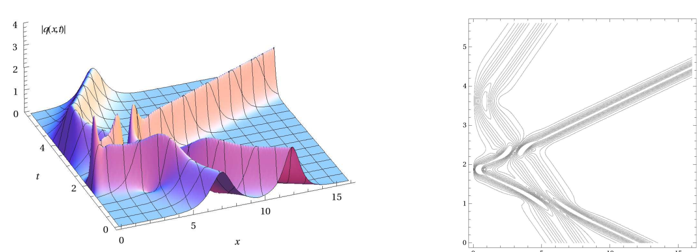

Figure 3: Reflection at the boundary of two physical solitons: $A_1 = 2$, $A_2 = 3/2$, $V_1 = -3$, $V_2 = -1$, $\xi_1 = 12$, $\xi_2 = 8$, and $\varphi_1 = \varphi_2 = 0$.

References

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[2] A S Fokas, *Phys. D* **35**, 167–185 (1989)

[3] A S Fokas, *Proc. Roy. Soc. London A* **453**, 1411–1443 (1997)