NEEDS 2009 Workshop
Nonlinear Evolution Equations and Dynamical Systems
Isola Rossa, May 16–23, 2009.

Bi-presymplectic representation of Liouville integrable systems and related separability theory

Maciej Błaszak[a]

February 24, 2009

a. Department of Physics, A. Mickiewicz University, Umultowska 85 , 61-614 Poznan, Poland.

1 Summary

Symplectic structures play an important role in the theory of Hamiltonian dynamical systems. In the case of a non-degenerate Poisson tensor the dual symplectic formulation of the dynamic can always be introduced via the inverse of the Poisson tensor. On the other hand, many dynamical systems admit Hamiltonian representation with degenerate Poisson tensor. For such tensors the notion of dual presymplectic structures was developed [4, 1, 2].

The alternative presymplectic picture is especially interesting in the case of Liouville integrable systems. There is well developed bi-Hamiltonian theory of such systems, bases on Poisson pencils of the Kronecker type [5, 6], with polynomial in pencil parameter Casimir functions and related separability theory (see [3], [7] and references quoted there in). The important question is whether it is possible to formulate an independent, alternative bi-presymplectic (bi-inverse-Hamiltonian in particular) theory of such systems with related separability theory and how both theories are related to each other.

The following presentation developes the bi-presymplectic theory of Liouville integrable systems. The whole formalism is based on the notion of *d-compatibility* of presymplectic forms and *d-compatibility* of Poisson bivectors.

First, we give some basic information on Poisson tensors, presymplectic two-forms, Hamiltonian and inverse Hamiltonian vector fields and dual Poisson-presymplectic pairs. Then, the concept of d-compatibility of Poisson bivectors and d-compatibility of closed two-forms is developed and the main properties of bi-presymplectic chains of arbitrary co-rank are investigated. We present conditions under which the bi-presymplectic chain is related to some Liouville integrable system and conditions when the chain is bi-inverse-Hamiltonian. We also present conditions under which Hamiltonian vector fields, constructed from a given bi-presymplectic chain, constitute a related bi-Hamiltonian chain. Finally we prove that arbitrary Stäckel system has bi-inverse-Hamiltonian formulation.

The advantage of bi-inverse-Hamiltonian representation compare to bi-Hamiltonian ones is that the existence of the first guarantee that related Liouville integrable system is separable and the construction of separation coordinates is purely algorithmic (in a generic case), while the bi-Hamiltonian representation does not guarantee the existence of quasi-bi-Hamiltonian representation

and hence separability of related system. Moreover, the projection of the second Poisson structure onto the symplectic foliation of the first one, in order to construct a quasi-bi-Hamiltonian representation, is fare from being trivial non-algorithmic procedure.

The general objects under investigation are bi-presymplectic chains of one-forms

$$\beta_i^{(k)} = \Omega_0 Y_i^{(k)} = \Omega_1 Y_{i-1}^{(k)}, \quad i = 0, 1, \ldots, n_k, \quad k = 1, ..., m, \tag{1}$$

where $n_1 + ... + n_m = n$, and (Ω_0, Ω_1) is a pair of d-compatible presymplectic forms of rank $2n$ and co-rank m. Each chain starts with a kernel vector field $Y_0^{(k)}$ of Ω_0 and terminates with a kernel vector field $Y_{n_k}^{(k)}$ of Ω_1. When $\beta_i^{(k)}$ are closed one-forms the chains are bi-inverse-Hamiltonian.

We prove that Stäckel separable systems with Hamiltonians given by separation relations of the most general form

$$\sum_{k=1}^{m} \varphi_i^k(\lambda_i, \mu_i) H^{(k)}(\lambda_i) = \psi_i(\lambda_i, \mu_i), \qquad i = 1, \ldots, n, \tag{2}$$

where

$$H^{(k)}(\lambda) = \sum_{i=1}^{n_k} \lambda^{n_k - i} H_i^{(k)}, \qquad n_1 + \cdots + n_m = n,$$

$\varphi_i^k(\lambda_i, \mu_i), \psi_i(\lambda_i, \mu_i)$ are arbitrary smooth functions and (λ, μ) are separation coordinates, have bi-inverse-Hamiltonian representation (1).

We also show how to construct separation coordinates from chains (1) in a purely algorithmic way.

References

[1] Błaszak M., *Presymplectic representation of bi-Hamiltonian chains*, J. Phys. A **37**, no. 50 (2004) 11971–11988

[2] Błaszak M. and Marciniak K., *Dirac reduction of dual Poisson-presymplectic pairs*, J. Phys. A **37**, no. 19 (2004) 5173–5187

[3] Błaszak M, *Degenerate Poisson Pencils on Curves: New Separability Theory*, J. Nonl. Math.Phys. **7** (2000) 213

[4] Dubrovin B. A., Giordano M., Marmo G. and Simoni A., *Poisson brackets on presymplectic manifolds,* Int. J. Mod. Phys. **8** (1993) 3747

[5] Gel'fand I. M. and Zakharevich I., *On the local geometry of a bi-Hamiltonian structure,* in: The Gel'fand Mathematical Seminars 1990-1992, eds. Corwin L. et. al., Birkhäuser, Boston 1993, p.51

[6] Gel'fand I. M. and Zakharevich I., *Webs, Lenard schemes, and the local geometry of bi-Hamiltonian Toda and Lax structures,* Selecta Math. (N.S) **6** (2000) 131

[7] Falqui G. and Pedroni M., *Separation of variables for bi-Hamiltonian systems, Math. Phys. Anal. Geom.* ***6*** *(2003) 139*