NEEDS 2009 Workshop
Nonlinear Evolution Equations and Dynamical Systems
Isola Rossa, May 16–23, 2009.

Symmetry reductions and exact solutions of a $K(m,n)$ equation with generalized evolution

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February 25, 2009

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1 Abstract

The methods of point transformations are a powerful tool in order to find exact solutions for nonlinear partial differential equations (PDE's). The classical theory of Lie point symmetries for differential equations describes the groups of infinitesimal transformations in a space of dependent and independent variables that leave the manifold associated with the equation unchanged [6]. The fundamental basis of this method is that, when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists. For PDE's with two independent variables a single group reduction transforms the PDE into ordinary differential equations (ODE's), which are generally easier to solve.

In the last years a great progress is being made in the development of methods and their applications to nonlinear ODE's for finding exact solutions [3, 4, 5, 7]. In this work we prove that given the following ODE

$$h'' + F\,(h')^2 + G = 0, \tag{1}$$

where $h' = \frac{dh}{dz}$, $F(h)$ and $G(h)$ are continuous functions, then for

$$G(h) = \alpha_1\, h^{\frac{2}{\beta}+1} + \alpha_2\, h^{\frac{-2}{\beta}+1} + \alpha_3\, F\, h^{\frac{2}{\beta}+2} + \alpha_4\, F\, h^{\frac{-2}{\beta}+2} + \alpha_5\, F\, h^2 + \alpha_6\, h, \tag{2}$$

and $F(h)$ an arbitrary function we obtain that:

• If

$$\begin{array}{lll} \alpha_1 = -\alpha^{\frac{-2}{\beta}}\,\beta\,(\beta+1)\,k, & \alpha_2 = -\alpha^{\frac{2}{\beta}}\,(\beta-1)\,\beta, & \alpha_3 = -\alpha^{\frac{-2}{\beta}}\,\beta^2\,k, \\ \alpha_4 = -\alpha^{\frac{2}{\beta}}\,\beta^2, & \alpha_5 = \beta^2\,(k+1)\,, & \alpha_6 = \beta^2\,(k+1)\,, \end{array} \tag{3}$$

$h(z) = \alpha\,\mathrm{sn}^\beta(z,k)$ is a solution of equation (1), where $\mathrm{sn}(z,k)$ is the Jacobi elliptic sine function [1] and α and β are parameters to be determined later.

• If

$$\begin{array}{lll} \alpha_1 = \alpha^{\frac{-2}{\beta}}\,\beta\,(\beta+1)\,k, & \alpha_2 = \alpha^{\frac{2}{\beta}}\,(\beta-1)\,\beta\,(k-1)\,, & \alpha_3 = \alpha^{\frac{-2}{\beta}}\,\beta^2\,k, \\ \alpha_4 = \alpha^{\frac{2}{\beta}}\,\beta^2\,(k-1)\,, & \alpha_5 = -\beta^2\,(2\,k-1)\,, & \alpha_6 = -\beta^2\,(2\,k-1)\,, \end{array} \tag{4}$$

$h(z) = \alpha \, \mathrm{cn}^{\beta}(z,k)$ is a solution of equation (1), where $\mathrm{cn}(z,k)$ is the Jacobi elliptic cosine function [1] and α and β are parameters to be determined later.

In the same way, we can obtain functions $G(h)$ for which functions $c(az+b)^n$, $a\exp(bz)$, $\mathrm{cd}(z,m)$, $\mathrm{sd}(z,m)$, $\mathrm{nd}(z,m)$, $\mathrm{dc}(z,m)$, $\mathrm{nc}(z,m)$, $\mathrm{sc}(z,m)$, $\mathrm{ns}(z,m)$, $\mathrm{ds}(z,m)$ and $\mathrm{cs}(z,m)$ (see [1]) are solutions of equation (1).

In the following we consider the $K(m,n)$ equation with generalized evolution term

$$u_t^l + au^m u_x + b(u^n)_{xxx} = 0, \tag{5}$$

where the first term is the generalized evolution term, while the second term represents the nonlinear term and the third term is the dispersion term. Also, $a, b \in \mathbb{R}$ are constants, while $l, m, n \in \mathbb{Z}^+$. This equation is the generalized form of the Korteweg-de Vries (KdV) equation, where, in particular, the case $l = m = n = 1$ leads to the KdV equation [2]. We make a classification of symmetry reductions of equation (5), depending on the values of the constants a, b, n, l and m, by using Lie theory. The symmetry reductions are derived from the optimal system of subalgebras and lead to ODE's.

Taking the travelling wave $u(x,t) = h(z)$, $z = \mu x - \lambda t$ into account we obtain from equation (5) the nonlinear ODE

$$h'' + \frac{n-1}{h}(h')^2 + \frac{a}{b\mu^2 n\,(m+1)} h^{m-n+2} - \frac{\lambda}{b\mu^3 n} h^{l-n+1} + \frac{k_1}{b\,\mu^3\,n}\, h^{1-n} = 0, \tag{6}$$

where k_1 is an integrating constant. By applying the result obtained for equation (1) to equation (6) we derive travelling wave solutions of equation (6) which yield to travelling wave solutions of equation (5). Among them we found a set of solutions with physical interest: solitons, kinks, antikinks and compactons.

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