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# Integrable Dynamics of Resonant Clusters: Evolution of Dynamical Phases

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## Abstract

It is well known that the dynamics of a Hamiltonian system depends crucially on whether or not it possesses nonlinear resonances. In the generic case, the set of nonlinear resonances consists of independent clusters of *resonantly interacting modes*, described by a few low-dimensional dynamical systems. We formulate and prove a new *theorem on integrability* which allows us to show that most frequently met clusters are described by integrable dynamical systems. Moreover we construct *explicit solutions for the so-called dynamical phases*, which are special combinations of the modes' phases chosen according to the resonance conditions. The results can be used directly for systems with cubic Hamiltonian.

**Resonant Clusters.** The simplest nonlinear resonant systems corresponding to the 3-wave resonance conditions are, in order of simplicity, triads (which are integrable) and small groups or clusters of connected triads which are known to be important for various physical applications: large-scale motions in the Earth's atmosphere [1], laboratory experiments with gravity-capillary waves [2], etc.

The triad is considered in the standard Manley-Rowe form:

$$\dot{B}_1 = ZB_2^* B_3, \quad \dot{B}_2 = ZB_1^* B_3, \quad \dot{B}_3 = -ZB_1 B_2, \tag{1}$$

where $(B_1, B_2, B_3)$ are complex amplitudes of 3 resonantly interacting modes $B_j \exp i(\mathbf{k}_j \cdot \mathbf{x} - \omega(\mathbf{k}_j)t)$, while the corresponding resonance conditions are

$$\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) - \omega(\mathbf{k}_3) = 0\,, \quad \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 = 0,$$

where $\omega(\mathbf{k})$ is the dispersion relation and $\mathbf{k}$ is the wavevector.

**Theorem on $(n-2)$–integrability.** *Let us assume that the autonomous system*

$$\frac{dy^j}{dt}(t) = \Delta^j(y(t)), \quad j = 1, \ldots, n$$

*possesses a standard Liouville volume density*

$$\sigma(y) : \ \sum_{j=1}^{n} \frac{\partial}{\partial y^j}(\sigma \Delta^j) = 0,$$

*and $(n-2)$ conservation laws (CL): $H_1(y), \ldots, H_{n-2}(y)$, functionally independent. Then a new CL in quadratures $H(y)$ can be constructed, which is functionally independent of the original ones, and therefore the system is integrable.*

We gave a constructive proof of this theorem for the case $n = 2$ in [3]. The knowledge of a canonical Hamiltonian structure *is not* required a priori. We applied this theorem in [3, 4] to prove integrability and also find explicit solutions of several dynamical systems arising in resonant interactions: triads ($n = 4$), kites ($n = 6$), and some cases of butterflies ($n = 7$). For simplicity we present here results only for triads.

If regarded in the amplitude-phase representation $B_j = C_j \exp i\theta_j$, Sys.(1) is equivalent to a system for the 3 real amplitudes $C_j$ and the **dynamical phase**, the phase combination $\varphi = \theta_1 + \theta_2 - \theta_3$:

$$\begin{cases} \dot{C}_1 = ZC_2C_3\cos\varphi, \quad \dot{C}_2 = ZC_1C_3\cos\varphi, \\ \dot{C}_3 = -ZC_1C_2\cos\varphi, \quad \dot{\varphi} = -(C_1^{-2} + C_2^{-2} - C_3^{-2})C_1C_2C_3\sin\varphi\,, \end{cases} \tag{2}$$

the individual phases $\theta_j$ being slave variables and obtainable by quadratures [5]. *Integrability* of dynamical system (2) is a well-known fact (e.g. [6]). Two of its CLs (the so-called Manley-Rowe relations) are: $H_1 = C_1^2 + C_3^2,\ H_2 = C_2^2 + C_3^2$. Sys.(2) has been used for a preliminary check of our method; in this case $n = 4$. The method can thus be applied and we obtain the following CL: $H = C_1C_2C_3\sin\varphi$, which turns out to be the Hamiltonian of original Sys.(1).

**Explicit solution for dynamical phase.** We established numerically in [4] the strong impact of dynamical phases on the behaviour of any physical system governed by a triad as well as small clusters of resonant triads. The analytical solution depends on the three real roots of a cubic polynomial in the so-called *Casus Irreducibilis*. Let $\rho = H_2/H_1$ and define $\alpha \in [0, \pi]$ by $\cos\alpha = \frac{(-2+3\rho+3\rho^2-2\rho^3)\,{H_1}^3 - 27H^2}{2\,(1-\rho+\rho^2)^{\frac{3}{2}}\,{H_1}^3}$. In terms of Jacobian elliptic functions with modulus $m = \cos\left(\frac{\alpha}{3} + \frac{\pi}{6}\right) / \cos\left(\frac{\alpha}{3} - \frac{\pi}{6}\right)$ and period $T = \frac{\sqrt{2}\,3^{\frac{1}{4}}\,\mathrm{K}(m)}{Z\,(1-\rho+\rho^2)^{\frac{1}{4}}\sqrt{\cos(\frac{\alpha}{3}-\frac{\pi}{6})}\,\sqrt{H_1}}$, the dynamical phase:

$$\varphi(t) = \cot^{-1}\left(\frac{m}{H}\left(\frac{2\mathrm{K}(m)}{Z\,T}\right)^3 \mathrm{sn\,cn\,dn}\left(2\,\mathrm{K}(m)\,\frac{(t-t_0)}{T}, m\right)\right)$$

evolves in the domain $(0, \pi)$, where $\mathrm{sn\,cn\,dn}(u, m) \equiv \mathrm{sn}(u, m)\,\mathrm{cn}(u, m)\,\mathrm{dn}(u, m)$.

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