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Orbifold Riemann surfaces and geodesic algebras

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1 Combinatorics of orbifold Riemann surfaces

We study the Teichmüller theory of Riemann surfaces with orbifold points of order two using the fat graph technique, which generalizes the fat graph technique by R.Penner and V.V.Fock. We obtain the surface $\Sigma_{g,s,|\delta|}$ of genus g, s holes ($s>0$) and $|\delta|\ge 0$ orbifold points of order two by factoring the Poincaré upper half plane $\mathbb{H}^2_+$ under the action of a Fuchsian group $\Delta_{g,s,|\delta|}\subset PSL(2,\mathbb{R})$. We then associate to this surface a fat graph $\Gamma_{g,s,|\delta|}$ of genus g with s faces, $|\delta|$ one-valent (pending) vertices and $4g-4+2s+|\delta|$ three-valent vertices. The total number of edges, $6g-6+3s+2|\delta|$, coincides with the dimension of the Teichmüller space of orbifold Riemann surfaces $\Sigma_{g,s,|\delta|}$.

We associate a real number Z_α to every (nondirected) edge of the graph $\Gamma_{g,s,|\delta|}$; the space $\mathbb{R}^{6g-6+3s+2|\delta|}$ of all these numbers is the Teichmüller space of orbifold Riemann surfaces. Morphisms of graphs $\Sigma_{g,s,|\delta|}$ correspond to three kinds of flip transformations of edges: the standard (Penner and Fock) flips of "inner" edges incident to three-valent vertices only, the "flips" on pending edges (that, generally, associate the new pending edge to another boundary component, or interchange pending edges inside the same boundary component), and changing of direction of spiraling of geodesic lines of ideal triangular decomposition to the given boundary component (hole). These transformations generate the complete modular, or braid, group in the case of algebras A_n and D_n below.

The main object of investigation are geodesic functions (traces of elements of the Fuchsian group $\Delta_{g,s,|\delta|}$, which are in one-to-one correspondence with closed paths in the graph $\Gamma_{g,s,|\delta|}$. They can be constructed following simple combinatorial rules: we set the matrix $\begin{bmatrix} 0 & -e^{Z_\alpha/2} \\ e^{-Z_\alpha/2} & 0 \end{bmatrix}$ every time we pass through the αth edge, the matrices of left and right turns, $L=\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ and $R=\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, every time we turn left or right at a three-valent vertex, and the matrix $F=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ of rotation through the angle π at an orbifold vertex.

2 Algebras A_n and D_n

The A_n algebras correspond to $\Sigma_{0,1,n}$—the disc with n orbifold points. The geodesic algebras coincide with algebras of monodromies of Dubrovin system of Fuchsian type $\frac{d\Psi(\sigma)}{d\sigma} = \sum_{i=1}^{n} \frac{E_i(-1/2-V)}{\sigma-u_i}\Psi(\sigma)$, $V^T = -V$, whose monodromies when going around square-root singularities u_i have the matrix form $(M_\beta)_{ij} = \delta_{i,j} - \delta_{\beta,i}G_{ij}$ with $G_{i,i} \equiv 2$. Then,

$$\mathrm{tr}\, M_i M_j = n - 4 + G_{i,j}^2,$$

and $G_{i,j}$ enjoy Poisson algebras of the Nelson–Regge–Ugaglia–Bondal type as well as the braid-group action, which can be presented in the matrix form as $\mathcal{A} \mapsto B_{i,i+1}\mathcal{A}B_{i,i+1}^T$, with $\mathcal{A}$ the upper-triangular $n \times n$ matrix with entries $G_{i,j}$ over the diagonal and units on the diagonal and with $B_{i,i+1}$ being the matrix with units on the diagonal and with the only nontrivial 2×2 block $\begin{bmatrix} G_{i,i+1} & -1 \\ 1 & 0 \end{bmatrix}$ on the intersection of ith and $(i+1)$th rows and columns.

We obtain D_n algebras from the system A_{n+s} (s can be arbitrarily large) by clashing s last poles of the original Fuchsian system. Introducing the special matrix $M_5 = M_{n+1}M_{n+2}\cdots M_{n+s}$, we consider the algebras of $G_{i,j}^{(k)}$, $i,j = 1,\dots,n$, where

$$\mathrm{tr}\, M_i M_5^{-k} M_j M_5^k = [G_{i,j}^{(k)}]^2 + n + s - 4,$$

and $G_{i,j}^{(k)} = G_{j,i}^{(-k)}$ turn out to be polynomials of the initial $G_{\alpha,\beta}$ with $\alpha,\beta = 1,\dots,n+s$. The Poisson and braid-group action on the set of $G_{i,j}^{(k)}$ is in general infinite, but there are two important reductions: the first, and basic one, is the level-p reduction when $(M_5)^p = I$. Then, $G_{i,j}^{(k+p)} = G_{i,j}^{(k)}$, and the action of the complete braid group has the finite-dimensional matrix form, $\lambda\mathbb{A} + \lambda^{-1}\mathbb{A}^T \mapsto \mathbb{B}_{i,i+1}(\lambda)(\lambda\mathbb{A} + \lambda^{-1}\mathbb{A}^T)\mathbb{B}_{i,i+1}(\lambda^{-1})^T$ with $\mathbb{A}$ being now the $(np)\times(np)$-matrix with blocks $\mathcal{A}^{(0)} = \mathcal{A}$ on the diagonal and $\mathcal{A}_{i,j}^{(k)} = G_{i,j}^{(k)}$ on the kth superdiagonal above the principal one.

The geometrical counterpart of D_n is the algebra of geodesic functions on the annulus $\Sigma_{0,2,n}$ with n orbifold points s_β. Then, $G_{i,j}^{(k)}$ corresponds to the geodesic that starts at s_i, winds k times around the selected hole, and terminates at s_j. It has self-intersections starting from $k = 1$ and $i \le j$, and the skein relations imply another reduction: $G_{i,j}^{(1)} + G_{j,i}^{(1)} = 2\overline{G}_{i,i}\overline{G}_{j,j} + (P^2 - 2)G_{i,j}^{(0)}$, where $\overline{G}_{i,i}$ are geodesic functions corresponding to going around the hole and s_i (having no analogues in the Fuchsian system description) and $P = e^\phi + e^{-\phi}$ is related to the hole perimeter ϕ. It is crucial that this reduction is Poissonic for *any* e^ϕ including the case where it is a pth root of unity. This is because neither braid-group action nor the Poisson algebras of $G_{i,j}^{(0)}$, $G_{i,j}^{(1)}$ ($i > j$), and $\overline{G}_{i,i}$ contain an explicit dependence on P. in particular, choosing $P^2 = 1$ corresponds to the level-3 reduction above and $P^2 = 2$ corresponds to the level-4 reduction. In the both cases (and for any level reduction with $p > 2$), we obtain the finite-dimensional representation of the braid-group action on the D_n algebras of geodesic functions on the surface $\Sigma_{0,2,n}$.

We also present the quantization of the Poisson relations and the quantum braid-group action and describe the central elements of A_n and D_n algebras in the cases of level-p and geometrical reductions.