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Integrable models in nonlinear optics and soliton solutions

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1 General introduction

The great variety of phenomena and effects which take place in nonlinear wave propagation, for instance phase self–modulation, resonant interactions, frequency conversion, polarization changes, shock formation among others, have motivated and produced quite a collection of mathematical models. No doubt, optics has been more prolific in this respect than any other branch of physics. Starting with the introduction of the Nonlinear Schrödinger equation around 50 years ago, our understanding of the interaction of electromagnetic radiation with matter, in both classical and quantum descriptions, has generated many, generally approximated, models in order to cope with the difficult problems related to nonlinearity. As it happens, quite a number of them, say more than ten, turn out to possess the important mathematical property of being *integrable*, a fact which makes nonlinear optics the proper stage for action of the theory of integrability and solitons. At least for some of these models, this very remarkable feature is not particularly surprising since they have been derived by a multiscale type reduction method and can be considered *universal*[1][2]. Be as it might be, this very fact, namely their integrability, strongly motivates a beneficial interaction between experts on integrable systems and the nonlinear optics community. Here a short introductory review of integrable models is given, with particular attention to wave-wave resonant interactions in connection to so-called boomeronic-type wave equations.

2 Recent results

To the purpose of introducing integrable multi component wave equations in 1+1 dimensions, the Lax pair formalism is set up in matrix notation, and the conditions for these equations to possess boomeron solutions is explicitly displayed. Notable examples of wave equations in this class are multi component Schroedinger type systems and resonant interaction models (in collaboration with F. Calogero [3][4]). Soliton solutions are then constructed by specializing the Dressing Darboux Transformation to deal with different boundary conditions which describe all–bright as well as mixed bright–dark and all–dark pulses (in collaboration with S. Lombardo [5][6]). Contact with nonlinear optics is

made by considering the simplest instance of boomeronic equation, namely the well–known three wave resonant interaction system. In this case only the mixed bright–dark soliton features, in the regime of two co–propagating bright pulses, a stable boomeron dynamics. The relevance of these solutions to propagation of laser pulses in chi2 crystals (in collaboration with F. Baronio, M Conforti and S. Wabnitz [7][8][9][10]) is briefly reported. A straight generalization of these results to the resonant interaction of five quasi–monochromatic waves as in the double–three–wave model will be mentioned. Open related mathematical problems will be finally pointed out in relation to the spectral theory of boomeronic type wave equations.

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