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Invariant solutions of the supersymmetric sine–Gordon equation

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The objective of this talk is to present a comprehensive group-theoretical symmetry analysis for two distinct forms of the supersymmetric sine–Gordon equation. First, we study a system of partial differential equations corresponding to the coefficients of the various powers of the fermionic independent variables. Next, we turn our attention to the super–sine–Gordon equation expressed in terms of a bosonic superfield involving fermionic independent variables. In each case, we use a generalization of the method of prolongation in order to determine the Lie (super)algebra of symmetries, and we present a systematic classification of all one-dimensional subalgebras of this resulting Lie (super)algebra. The method of symmetry reduction then allows us to derive invariant solutions of the supersymmetric sine–Gordon model. Several new types of algebraic, hyperbolic and doubly periodic solutions are obtained in explicit form.

Background

The $(1+1)$-dimensional sine–Gordon equation

$$u_{xt} = \sin u. \tag{1}$$

has applications in various areas of physics (see e.g. [1, 2] and references therein) and also has great significance in mathematics, especially in the soliton theory of surfaces [2, 3, 4]. More recently, a supersymmetric extension has been established for the sine–Gordon equation (1) and a number of significant results have been determined [5, 6, 7, 8]. For instance, it has been shown [6] that the equation of motion appears as the compatibility condition of a set of Riccati equations.

The supersymmetric sine–Gordon equation is constructed on the 4-dimensional Grassmannian superspace $\{(x, t, \theta_1, \theta_2)\}$. Here, the variables x and t represent the bosonic coordinates of 2-dimensional Minkowski space, while the quantities

θ_1 and θ_2 are anticommuting fermionic variables. The bosonic function $u(x,t)$ is replaced by the scalar bosonic superfield

$$\Phi\left(x,t,\theta_1,\theta_2\right) = \frac{1}{2}u(x,t) + \theta_1\phi(x,t) + \theta_2\psi(x,t) + \theta_1\theta_2 F(x,t), \tag{2}$$

where ϕ and ψ are fermionic-valued fields and F is a bosonic field. The supersymmetric extension of the sine–Gordon equation (1) is constructed in such a way that it is invariant under the supersymmetry generators

$$Q_x = \partial_{\theta_1} - \theta_1\partial_x \qquad \text{and} \qquad Q_t = \partial_{\theta_2} - \theta_2\partial_t. \tag{3}$$

This is ensured by writing the supersymmetric sine–Gordon equation is terms of the covariant derivative operators

$$D_x = \partial_{\theta_1} + \theta_1\partial_x \qquad \text{and} \qquad D_t = \partial_{\theta_2} + \theta_2\partial_t, \tag{4}$$

which possess the property that they anticommute with the supersymmetry generators (3). The supersymmetric sine–Gordon equation is given by the equation

$$D_x D_t \Phi = \sin\Phi. \tag{5}$$

The Lie superalgebra of the superfield form of the supersymmetric sine–Gordon equation (5) is spanned by the following five infinitesimal symmetries: two translations (in the x and t directions), one scaling transformation involving the independent variables, and the two supersymmetry transformations (3).

References

[1] G. L. Lamb Jr., *Elements of Soliton Theory* (Wiley, New York, 1980).

[2] C. Rogers and W. K. Schief, *Bäcklund and Darboux Transformations* (Cambridge University Press, Cambridge, 2002).

[3] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).

[4] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Cambridge, 1991).

[5] B. Grammaticos, A. Ramani and A. S. Carstea, "Bilinearization and soliton solutions of the $N=1$ supersymmetric sine-Gordon equation," *J. Phys. A: Math. Gen.* **34**, 4881-4886 (2001).

[6] M. Siddiq and M. Hassan, "On the linearization of the super sine-Gordon equation," *Europhys. Lett.* **70**, 149-154 (2005).

[7] M. Siddiq, M. Hassan and U. Saleem, "On Darboux transformation of the supersymmetric sine-Gordon equation," *J. Phys. A* **39**, 7313-7318 (2006).

[8] Q. P. Liu and M. Manas, "Pfaffian solutions for the Manin-Radul-Mathieu SUSY KdV and SUSY sine-Gordon equations," *Phys. Lett. B* **436**, 306-310 (1998).