NEEDS 2009 Workshop
Nonlinear Evolution Equations and Dynamical Systems
Isola Rossa, May 16–23, 2009.

Squared Eigenfunctions for 3×3 Eigenvalue Problems

D.J. Kaup[a,b]

March 2, 2009

a. Mathematics Department, PO Box 161364, University of Central Florida, Orlando, FL 32816-1364.

b. Institute for Simulation and Training, 3100 Technology Parkway, Suite 100, University of Central Florida, Orlando, FL 32826.

1 General remarks

Integrable equations in one dimension have Lax pairs and Lax pairs consists of a spatial eigenvalue problem with a spectral parameter and another eigenvalue problem which is the "spectral evolution equation". These operators operate on the same functions so one has a single function which has to satisfy two independent conditions. These functions are called Jost functions. One operator evolves the Jost function spatially and the other which evolves it in time (or another dimension). Thus integrability conditions must be satisfied in order for nontrivial Jost functions to exist. These integrability conditions are called "integrable evolution equations" and they are a set of nonlinear conditions on the evolution of the various potential-like components contained in the members of the Lax pair.

The spatial eigenvalue problem determines what the scattering data is and defines what the inverse scattering problem must be. Any given spatial eigenvalue problem can also have a hierarchy of spectral evolution equations with which it could be paired as a Lax pair. For each of these pairs, one would have a different integrable evolution equation. For any given Lax pair, it could also have several possible reductions, wherein various components of the potentials would be identified. On the other hand, there is always the most general spectral eigenvalue problem where there are no reductions applied and all potential components are taken to be uniquely different. For example, the AKNS eigenvalue problem [1] is the most general spectral eigenvalue problem for the 2×2 Dirac case, since the potential matrix has only two nonzero, nontrivial components, q and r, each of which are taken to be unrelated. An example of a reduction for this problem would be to take $r = q$.

These points are important when one considers perturbations of integrable nonlinear evolution equations. There are two ways in which one could approach perturbations. First, one could slightly shifts the initial data of the potentials. Second, one could add additional terms to the integrable evolution equations, causing it to become non-integrable, although close in some sense to an integrable case. In the first approach, one would want to determine how the scattering data would shift when the initial values of the potentials were slightly shifted. The functions which provide this mapping are called the "adjoint squared eigenfunctions" (ASE). The inverse of this mapping provides how the potentials shift when the scattering data is slightly shifted. The functions which provide this mapping are called the "squared eigenfunctions" (SE). In the second case, one has the additional problem of determining how the additional terms which are added to the integrable evolution equations shift the evolution of the scattering data. Both these approaches would still use the same methodology for mapping between the shifts in the scattering data and the shifts in the potentials.

Methods for obtaining the SE and the ASE are important for any perturbation studies of integrable systems. The original method for obtaining these [2, 3] was rather tedious and long. Since that time, much more has been understood about the SE due to the efforts of Gerdjikov [4], Yang[5] and others [6]. Recent work by Yang and myself [7, 8], along with excellent hindsight, has allowed us to back off and take a broader view of this problem, and as a result, simplify the problem of constructing the SE and the ASE into a set of well defined actions. The differences from one eigenvalue problem to another would be expected to show up only as differences in the exact mechanics as to how one executes each action. This approach will be illustrated with the AKNS system and the 3×3 eigenvalue problem for the Sasa-Satsuma equation. If time allows, the general 3×3 eigenvalue problem, without any reductions, will be discussed.

References

[1] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, "The inverse scattering transform- Fourier analysis for nonlinear problems", Studies in Appl. Math. **LIII**, 249-315 (1974).

[2] D. J. Kaup, A perturbation expansion for the Zakharov-Shabat inverse scattering transform, SIAM J. Appl. Math. **31**, 121 (1976).

[3] D. J. Kaup, Closure of the squared Zakharov-Shabat eigenstates, J. Math. Anal. Appl. **54**, 849864 (76).

[4] Gerdjikov V. S., "Generalized Fourier Transforms for the Soliton Equations, Gauge Covariant Formulation", Inverse Problems **2**, 51-74 (1986).

[5] Jianke Yang, "Eigenfunctions of Linearized Integrable Equations Expanded Around an Arbitrary Solution", Stud. Appl. Math. **108**, 145-59 (2002)

[6] V.I. Karpman and E.M. Maslov, "Perturbation theory for solitons", Zh. Eksp. Teor. Fiz. **73**, 537-559 (1977).

[7] Jianke Yang & D. J. Kaup, "Derivation of Squared Eigenfunctions for the Sasa-Satsuma Equation", J. Math. Phys. **50**, 023504 (2009).

[8] D. J. Kaup, "Integrable Systems and Squared Eigenfunctions". (Submitted to the Proceedings of the workshop "Nonlinear Physics: Theory and Experiment. V", Gallipoli, Italy, June 12-21, 2008, which will be published as special issues of Theoretical and Mathematical Physics.)