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Coisotropic deformation of an elliptic curve and linear flows on the first Birkoff stratum

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Introduction

In [3] the authors remark the relevance of the coisotropic deformations of associative commutative algebras in the context of the dispersionless hierarchies of integrable systems. The starting point of this approach is an n-generated polynomial algebra $\mathbb{C}[p_0, \dots, p_{n-1}]$ and an ideal $\mathcal{J}_0 \subset \mathbb{C}[p_0, \dots, p_{n-1}]$. Let us next consider a deformation $\mathcal{A}_x^n(p)$ of the algebra given by an n-generated algebra of polynomials in $p_0, \dots, p_{n-1}$ whose coefficients depend on the parameters $x_0, \dots, x_{n-1}$. In the same way let us also take a deformation $\mathcal{J}_x^n(p) \subset \mathcal{A}_x^n(p)$ of the ideal $\mathcal{J}_0$.
The key point in the definition of coisotropic deformation is considering the elements of $\mathcal{A}_x^n(p)$ as functions of a suitable Poisson space, in our cases $\mathbb{R}^{2n}$ with co-ordinates (x_i, p_j) and equipped with the canonical structure $\{x_i, p_j\} = \delta_{ij}$.
The algebra $\mathcal{A}_x^n(p) \backslash \mathcal{J}_x^n(p)$ is a *coisotropic deformation* of $\mathbb{C}[p_0, \dots, p_{n-1}] \backslash \mathcal{J}_0$ if

$$\{\mathcal{J}_x^n(p)\,,\, \mathcal{J}_x^n(p)\} \subset \mathcal{J}_x^n(p). \tag{1}$$

In this setup the dispersionless Kadomtsev-Petviashvilii equation is related to a coisotropic deformation [3] of the polynomial algebra $\mathbb{C}[p_1, p_2, p_3]$ with the ideal $J_{KP} = \langle p_3 - p^3 - u_1 p - u_0 \ ,\ p_2 - p^2 - v_0 \rangle$. If we introduce the deformation $u_i = u_i(x_1, x_2, x_3)$ for $i = 0, 1$ and $v_0 = v_0(x_1, x_2, x_3)$, then the condition

$$\{p_3 - {p_1}^3 - u_1 p_1 - u_0 \ ,\ p_2 - {p_1}^2 - v_0\} \in J_{KP}$$

gives rise to the dKP equation. Other similar $2+1$D integrable systems such as n-dKP, can be interpreted as coisotropic deformations of a polynomial algebra quotiented by an ideal generated by algebraic curves of genus zero.

Our results

In [5] we extend this approach taking into account ideals generated by genus 1 algebraic curves. We start from $\mathbb{C}[p_2, p_3, p_4]$ and the ideal $\mathcal{J}_1 = \langle \mathcal{E}, J^{(4)} \rangle$, where

$$J^{(4)} = p_4 - {p_2}^2 - v_2 p_3 - v_1 p_2 - v_0$$

and

$$\mathcal{E} = {p_3}^2 - {p_2}^2 - u_4 p_3 p_2 - u_3 {p_2}^2 - u_2 p_3 - u_1 p_2 - u_0$$

is a generic elliptic curve. We show that the coisotropy conditions

$$\{\mathcal{E}, J^{(4)}\} \in \mathcal{J}_1, \tag{2}$$

applied to the algebra deformation depending on the parameters x_2, x_3, x_4, are equivalent to the dispersionless counterpart of the $2+1$D linear flow on the first Birkoff stratum of the Universal Sato Grassmannian (see for this system [1, 4]). In general, we conjecture that the dispersionless counterpart of any Grassmannian integrable flow [7, 2] are deformed associative algebras. The complete classification of the algebras and the related tau structure is a work in progress. In the literature, e.g. in [6], the authors show that the dispersionless 3 component KdV equation can be interpreted as a $1+1$D deformation of an elliptic curve: Such deformation is a particular reduction of the $2+1$D system obtained from (2).
The extension of all the previous results to the whole hierarchy implies the use of an infinitely many generated associative algebra.

References

[1] P. Casati, B. G. Konopelchenko, M. Pedroni, *A bi-Hamiltonian approach to hidden KP equations*, VI Workshop on Partial Differential Equations, Part I (Rio de Janeiro, 1999) Mat. Contemp. 18 (2000), 59–75

[2] G. Falqui, F. Magri, M. Pedroni, *Bi-Hamiltonian geometry, Darboux coverings, and linearization of the KP hierarchy*, Comm. Math. Phys. 197 (1998), no. 2, 303–324

[3] B. G.Konopelchenko, F. Magri, *Coisotropic deformations of associative algebras and dispersionless integrable hierarchies*, Comm. Math. Phys. 274 (2007), no. 3, 627–658

[4] B. G. Konopelchenko, L. M. Alonso, Luis Martnez, E. Medina, *Singular sector of the Kadomtsev-Petviashvili hierarchy, $\overline{\partial}$ operators of nonzero index, and associated integrable systems*, J. Math. Phys. 41 (2000), no. 1, 385–413

[5] B. G. Konopelchenko, G. Ortenzi, *Coisotropic deformations of elliptic curves*, in preparation

[6] Y. Kodama, B. G. Konopelchenko, *Singular sector of the Burgers-Hopf hierarchy and deformations of hyperelliptic curves*, J. Phys. A 35 (2002), no. 31, L489-L500

[7] G. Segal, G. Wilson, *Loop groups and equations of KdV type*, Inst. Hautes Études Sci. Publ. Math. No. 61 (1985), 5–65