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The Generalized Symmetry Method for Discrete Equations

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The discovery of new two-dimensional integrable partial difference equations (or $\mathbb{Z}^2$-lattice equations) is always a very challenging problem as, by proper continuous limits, many other results on integrable differential-difference and partial differential equations may be obtained.

The classification of integrable nonlinear partial differential equations has been widely discussed. For example in the classification scheme introduced by Shabat, where the formal symmetry approach has been introduced (see [6] for a review). This approach has been successfully extended to the differential-difference case by Yamilov [11]. In the completely discrete case the situation turns out to be quite different, and, up to now, the formal symmetry technique has not been able to provide any result. In the case of difference–difference equations, the first exhaustive classification has been obtained in [1] by Adler and in [2] by Adler, Bobenko and Suris for linear affine equations. The so obtained equations have been thoroughly studied by many researchers, and it has been shown that they have Lax pairs and that possess generalized symmetries [7, 5, 9].

We study here the class of autonomous discrete equations on the lattice $\mathbb{Z}^2$:

$$u_{i+1,j+1} = F(u_{i+1,j}, u_{i,j}, u_{i,j+1}), \tag{1}$$

where i, j are arbitrary integers. Many integrable examples of equations of this form are known [2, 3, 4, 10, 9]. Requiring additional geometrical symmetry properties, a classification result has been obtained in [2] together with a list of integrable equations. However, the symmetries for those discrete equations obtained in [7, 9] show that the obtained class of equations contained in [2] is somehow restricted [5]. From [5] it follows that one should expect a larger number of integrable discrete equations of the kind of eq. (1) than those up to now known.

Eqs. (1) are possible discrete analogs of the hyperbolic equations

$$u_{x,y} = F(u_x, u, u_y). \tag{2}$$

Eqs. (2) are very important in many fields of physics, and, as such, they have been studied using the generalized symmetry method, however without much success. Only the following two particular cases:

$$u_{x,y} = F(u), \tag{3}$$

$$u_x = F(u,v), \quad v_y = G(u,v), \tag{4}$$

which are essentially easier, have been solved [12, 13]. The study of the class of equations (1) may be important to characterize the integrable subcases of eq. (2).

Here, following the standard scheme of the generalized symmetry method, we derive a few integrability conditions for the class (1). These conditions are not sufficient to carry out a classification of the discrete equations (1). To be concrete we reduce ourselves to consider just 5 points generalized symmetries. This request provides further integrability conditions. With these extra conditions, the set of obtained conditions will be suitable for testing and classifying simple classes of difference equations of the form (1). As an example we apply those conditions to the class of equations

$$u_{i+1,j+1} = u_{i+1,j} + u_{i,j+1} + \varphi(u_{i,j}), \tag{5}$$

a trivial approximation to the class (3). This calculation is an example of classification problem for a class depending on one unknown function of one variable. This class contains trivial approximations of some well-known integrable equations included in the class (3), namely, the sine-Gordon, Tzitzèika and Liouville equations.

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