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Algebraic solutions of the sixth Painlevé equation

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Main topic of this communication is the sixth Painlevé equation (see [4]):

$$\frac{d^2w}{dt^2} = \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}\right)\left(\frac{dw}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{w-t}\right)\frac{dw}{dt} + \\ + \frac{w(w-1)(w-t)}{2t^2(t-1)^2}\left((\theta_\infty-1)^2 - \frac{\theta_x^2 t}{w^2} + \frac{\theta_y^2(t-1)}{(w-1)^2} + \frac{(1-\theta_z^2)t(t-1)}{(w-t)^2}\right).$$

This is the most general ODE of the form $w'' = F(t, w, w')$, with F rational in w, w' and t, whose general solution has no movable branch points and essential singularities.

A result from Watanabe [8] suggests that, roughly speaking, any solution of Painlevé VI is either a) algebraic or b) solves a Riccati equation or c) cannot be expressed via classical functions. This raises the problem of construction and classification of algebraic solutions. Known examples turn out to be related to various mathematical structures, including e.g. Frobenius manifolds [2], symmetry groups of regular polyhedra [3], complex reflections [1], Grothendieck's dessins d'enfants and their deformations [5]. In the case $\theta_x = \theta_y = \theta_z = 0$, a full classification of algebraic solutions has been obtained by Dubrovin and Mazzocco [3].

Painlevé VI can also be seen as the equation of monodromy preserving deformation of Fuchsian systems with 4 regular singular points. Analytic continuation of its solutions induces an action of the pure braid group $\mathcal{P}_3$ on the space of monodromy data of these systems (the quotient $\mathcal{M} = G^3/G$, $G = SL(2,\mathbb{C})$). Since any algebraic solution has a finite number of branches, the corresponding braid group orbit should be finite. Using this idea (suggested in [3] and going back to Schwarz's studies [7] of the hypergeometric equation), we classify all algebraic Painlevé VI solutions:

Theorem. *Up to parameter equivalence, the list of all Painlevé VI solutions with finite branching consists of the following:*

- *an infinite family of algebraic Picard and Riccati solutions;*
- *three algebraic families depending on continuous parameters;*
- *45 exceptional algebraic solutions.*

The communication will be based on the results of [6].

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