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Bilinear discretization of quadratic vector fields

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1 The problem of integrable discretization

This talk deals with some aspects of the problem of integrable discretization, as defined in [1]. Consider a completely integrable flow

$$\dot{x} = f(x) = \{H, x\}, \tag{1}$$

with a Hamilton function H on a Poisson manifold $\mathcal{P}$ with a Poisson bracket $\{\cdot,\cdot\}$. Thus, the flow (1) possesses sufficiently many functionally independent integrals $I_k(x)$ in involution. The problem consists in finding a family of diffeomorphisms $\Phi : \mathcal{P} \to \mathcal{P}$,

$$\tilde{x} = H\Phi(x;\epsilon), \tag{2}$$

depending smoothly on a small parameter $\epsilon > 0$, with the following properties:

1. The maps (2) approximate the flow (1): $\mathrm{H}\Phi(x;\epsilon) = x + \epsilon f(x) + O(\epsilon^2)$.
2. The maps (2) are Poisson w. r. t. the bracket $\{\cdot,\cdot\}$ or some its deformation $\{\cdot,\cdot\}_\epsilon = \{\cdot,\cdot\} + O(\epsilon)$.
3. The maps (2) are integrable, i.e. possess the necessary number of independent integrals in involution, $I_k(x;\epsilon) = I_k(x) + O(\epsilon)$.

2 Hirota-Kimura-Kahan type discretization of quadratic vector fields

The talk will be devoted to discretizations of the type introduced in [2, 3] and missing from the book [1], despite its encyclopedic nature. Reasons for this omission: discretization of the Euler top [2] seemed to be an isolated curiosity; discretization of the Lagrange top [3] seemed to be incomprehensible, if not even wrong. It turns out that the discretizations of Hirota-Kimura are instances of a general method for discretizing differential equations with quadratic vector fields, proposed by W. Kahan in [4]. It is applicable to any system of ordinary differential equations for $x : \mathbb{R} \to \mathbb{R}^n$ with a quadratic vector field

$$\dot{x} = Q(x) + Bx + c,$$

where $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a quadratic function, while $B \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ and $c \in \mathbb{R}^n$. Kahan's discretization reads as

$$\frac{\tilde{x} - x}{\epsilon} = Q(x, \tilde{x}) + B(x + \tilde{x}) + c,$$

where $Q(x, \tilde{x}) = Q(x + \tilde{x}) - Q(x) - Q(\tilde{x})$, is the symmetric bilinear function corresponding to Q. General features of this discretization:

1. discrete equations are linear w.r.t. $\tilde{x}$ and define therefore an explicit (rational) map $\tilde{x} = f(x, \epsilon)$;

2. this map is reversible (therefore birational): $f^{-1}(x, \epsilon) = f(x, -\epsilon)$.

Kahan illustrated his method with an application to the famous Lotka-Volterra system, where it produces non-spiralling orbits, unlike the majority of conventional integrators.

A bi-Hamiltonian structure of the Hirota-Kimura discretizations of the Euler top was established in [5]. The Kahan's procedure has been then used to obtain an integrable discretization of the Clebsch system [6] and of some three bi-Hamiltonian flows [7]. An overview of recent results will be presented.

References

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