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Differential operators on the space of periodic functions

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The problem of the equivalence of linear differential operators, notion first introduced by Delsarte (1938), is a long standing one. Two operators A and B on a space H are equivalent if there is an isomorphism X such that $AX = XA$.

On the space $C_{2\pi}^{\infty}(R)$ of all $2\pi-$periodic $C^{\infty}-$functions on R, we consider two finite linear differential operators $T_1 = \sum_{k=1}^{m} a_k D^k$ and $T_2 = \sum_{k=1}^{m} b_k D^k$, where a_k, b_k are either constants or exponential functions $e^{ipx}, p \in Z$.

It is well-known that the space $C_{2\pi}^{\infty}(R)$ with its natural topology given by the seminorms

$$\|f\|_k^2 = \sum_{0 \le p \le k} \left\| f^{p)} \right\|_{L_{2[-\pi,\pi]}}^2$$

is isomorphic to the sequence space s of rapidly decreasing sequences

$$s = \left\{ x = (x_n) : \lim |x_j| \, j^k = 0, \text{for all } k \in N \right\}$$

with its natural topology given by the seminorms

$$\|x\|_k = \sum_{j \in N} |x_j| \, j^k \text{ for all } k \in N$$

that is the mapping

$$f \xrightarrow{F} (f_0, f_1, f_{-1}, f_2, f_{-2}, ...)$$

is a linear bijection, where

$$f(x) = \sum_{n \in Z} f_n e^{inx}$$

is the Fourier series of f [6].

The operators T_1 and T_2 induce two linear operators A_1 and A_2 from s to s, in fact

$$\begin{aligned} A_1 &= F T_1 F^{-1} \\ A_2 &= F T_2 F^{-1} \end{aligned}$$

The problem addressed in this work is the following one: Is there an isomorphism S such that

$$SA_1 = A_2 S$$

and consequently

$$F^{-1}SFT_1 = T_2F^{-1}SF$$

which implies that the operators T_1 and T_2 are equivalent?.

The answer to this problem in our particular case seems to be negative, at least the examples we present do not have a solution. We intend to go more deeply into the problem in future works.

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